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Free Products with Amalgamation of Inverse Semigroups

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1. INTRODUCTION AND SUMMARY

Inverse semigroups are (to within isomorphism) precisely the semigroups of partial one-to-one transformations of sets that with each transformation contain also the inverse of that transformation [1, Section 1.9], just as the concept of a group is an abstract characterization of those semigroups of permutations of sets that with each permutation contain also the inverse of that permutation. The object of this paper is to prove that the class of inverse semigroups has the strong amalgamation property, or equivalently, that in the free inverse semigroup product $\bigtimes_U^* S_i$ with amalgamation over U of a set $\{S_i : i \in I\}$ of inverse semigroups each with a common inverse subsemigroup U, "no collapse occurs." Two subsidiary results proved are of considerable interest: let T be any inverse subsemigroup of any inverse semigroup S; then any right congruence on T extends in a special way to a right congruence on S; and any representation of T by one-to-one partial transformations extends to one of S, in a certain, strong sense.

The specialization of our proof to groups appears to give a new proof of the long-known corresponding result for groups; see the literature in Neumann [5], dating back to Schreier [7] in 1927. The amalgamation property for arbitrary semigroups was first extensively studied by Howie [1, Chap. 9]. Preston [6, p. 136] points out that Howie and Isbell's proof that inverse semigroups are absolutely closed [2, Theorem 2.3] also gives a proof that the class of inverse semigroups has what we shall define below as the special amalgamation property ("$S *_U S$ does not collapse" for any inverse subsemigroup U of any inverse semigroup S); he also poses there the question of whether or not inverse semigroups have the strong amalgamation property.

Being of the form "a class of semigroups (in fact a variety of algebras) has the strong amalgamation property," our result has many interesting consequences. Jónsson [3] gives many consequences (too numerous to repeat here) of the (weak) amalgamation property for classes of structures, most of which are new results for inverse semigroups; the result [3, p. 150] that any group G can be embedded in a group G' such that isomorphisms between subgroups of G are induced by inner automorphisms of G' also extends to inverse semigroups (by the *inner automorphism* i_a of an inverse semigroup S with identity 1 induced by a unit $a \in S$ we mean the automorphism which maps each $x \in S$ to $a^{-1}xa$). Further, many of the results proved in [5] from the amalgamation property for groups will now extend to inverse semigroups. The author will make these results explicit in a further paper.

2. Preliminaries

We use wherever possible, and usually without comment, the notations and conventions of Clifford and Preston [1].

A class of algebras $\mathscr{A}$ is said to have the *strong amalgamation property* if for any indexed set of algebras $\{A_i: i \in I\}$ from $\mathscr{A}$, each having an algebra $U \in \mathscr{A}$ as a subalgebra, there exist an algebra $B \in \mathscr{A}$ and monomorphisms $\phi_i: A_i \to B$ (for each $i \in I$) such that

(i) $\phi_i|_U = \phi_j|_U$, for all $i, j \in I$

(ii) $(A_i\phi_i) \cap (A_j\phi_j) = U\phi_i$, for all $i, j \in I$ with $i \neq j$.

Omitting the condition (ii) gives us the definition of the *weak amalgamation property*. Adding the condition that $A_i = A_j$ for all $i, j \in I$, to the hypothesis of the definition of the strong amalgamation property gives us the definition of the *special amalgamation property*.

It is well-known (see [3, Section 3] for weak amalgamation) and easy to show that in a class of algebras closed under isomorphisms and the formation of the union of any ascending chain of algebras, each amalgamation property follows from the case in which $|I| = 2$, by transfinite induction. Hence we shall consider in this paper only the case $|I| = 2$.

Clearly, in a class of algebras closed under isomorphisms, the weak and the special amalgamation properties together imply the strong amalgamation property. Rather than prove the weak amalgamation property for inverse semigroups and then appeal to the special property, we shall prove the strong amalgamation property directly.

The faithful representation $\rho: S \to \mathscr{I}_S$ of any inverse semigroup S by one-to-one partial transformations, constructed in [1, Section 1.9], we shall call the Preston–Vagner representation of S; recall then that for each $s \in S$, $\rho_s: Ss^{-1} \to Ss$, and for each $x \in Ss^{-1}$, $x\rho_s = xs$.

For an arbitrary representation $\rho: S \to \mathscr{I}_X$ of an inverse semigroup S and for any inverse subsemigroup U of S, $\rho|_U$ denotes the representation of U

induced by ρ, i.e., $(\rho\,|_U)\colon U \to \mathscr{I}_X$ and $(\rho\,|_U)_u = \rho_u$ for all $u \in U$. Further, for any subset Y of X, $(\rho\,|_U) \mid Y$ denotes the mapping of U into $\mathscr{I}_X$ which maps each $u \in U$ to $\rho_u\,|_Y$, the partial transformation ρ_u with its domain restricted to $Y \cap (\text{Domain } \rho_u)$.

3. An Example

We give an example, due to C. J. Ash, which shows that the class of finite inverse semigroups does not have even the weak amalgamation property. This is to justify, to some extent, the infinite procedure in our later construction. Other classes are mentioned at the end of the paper.

Let S and S' be semigroups both isomorphic to the five element Brandt semigroup $\mathscr{M}^0(\langle 1\rangle; 2, 2; \Delta)$. Denote the zero of each of S and S' by 0 and denote the nonzero idempotents of S and S' by e, f and e, g respectively. Extend the multiplication of S' to one of $T = S' \cup \{f\}$ by defining $f^2 = f$, $fx = gx$, $xf = xg$, for all $x \in S'$. Then T is an inverse semigroup, $S \cap T \supseteq \{0, e, f\} = U$ say, and U is a subsemilattice of both S and T.

Now suppose we have monomorphisms $\phi\colon S \to W$ and $\psi\colon T \to W$ of S and T, respectively, into any semigroup W such that $\phi \mid U = \psi \mid U$. Then $g < f$, $g \mathscr{J} e$ (in T) and $e \mathscr{J} f$ (in S) so $g\psi < f\psi$ and $g\psi \mathscr{J} e\psi = e\phi \mathscr{J} f\phi = f\psi$ (in W). Hence W contains a copy of the bicyclic semigroup [1, Theorem 2.54], and is infinite.

4. Right Congruences

Theorem 1. *Let U be any inverse subsemigroup of any inverse semigroup T and let δ be any right congruence on U. Then δ extends to a right congruence on T whose restriction to U is δ. Hence if we put $\gamma = \delta^*$, the right congruence on T generated by δ, then $\gamma \cap (U \times U) = \delta$.*

Proof. Put $\delta_1 = \{(au, bu) \in T \times T\colon (a, b) \in \delta, u \in T^1\} \cup \iota_T$. Then $\delta^* = \delta_1{}^t = \bigcup_{n=1}^{\infty} \delta_1{}^n$. We make the convention that $\delta_1{}^0 = \iota_T$. Then $\delta^* = \bigcup_{n=0}^{\infty} \delta_1{}^n$ also. Since $\delta \subseteq \delta^* \cap (U \times U)$ it remains to show the following:

For $n = 0, 1, 2,\ldots$, for all $a, b \in U$, if $(a, b) \in \delta_1{}^n$, then $(a, b) \in \delta$.

We use induction on n. For $n = 0$ the statement is trivial. So let us take any integer $n > 0$ and assume the statement for $n - 1$ is true. Take any pair $(a, b) \in \delta_1{}^n \cap (U \times U)$. Then for some elements $x, y \in T$,

$$a\delta_1 x \delta_1^{n-1} b, \qquad a\delta_1^{n-1} y \delta_1 b.$$

Since δ_1 and hence δ_1^{n-1} are right compatible, we can obtain

$$a = aa^{-1}a\delta_1 xa^{-1}a\delta_1^{n-1}ba^{-1}a, \tag{1}$$

$$ba^{-1}a = bb^{-1}ba^{-1}a\delta_1 yb^{-1}ba^{-1}a\delta_1^{n-1}ab^{-1}ba^{-1}a = ab^{-1}b, \tag{2}$$

$$ab^{-1}b\delta_1^{n-1}yb^{-1}b\delta_1 bb^{-1}b = b. \tag{3}$$

The following lemma, and the inductive hypothesis, applied to each of the statements (1), (2), and (3), will give us that $(a, b) \in \delta$.

LEMMA 2. *For any $c \in U$ and $d \in T$, if $c\delta_1 dc^{-1}c$ then $dc^{-1}c \in U$ and $c\delta dc^{-1}c$.*

Proof. If $c = dc^{-1}c$ we are there, so assume $c \neq dc^{-1}c$. Then for some $(x, y) \in \delta$ and $u \in T^1$, we have

$$c = xu, \qquad yu = dc^{-1}c.$$

Then

$$\begin{aligned} dc^{-1}c &= dc^{-1}cc^{-1}c = yu(xu)^{-1}(xu) \\ &= yuu^{-1}x^{-1}xu = yx^{-1}xuu^{-1}u = yx^{-1}c \in U. \end{aligned}$$

Also $dc^{-1}c = yx^{-1}c\delta xx^{-1}c = xx^{-1}xu = c$, giving the lemma.

Remark 1. The theorem does not extend to arbitrary semigroups, in fact not even to bands (e.g., put $U = \{f, g, h\}$, a right zero semigroup, and $T = U \cup \{e\}$, with $ex = x$ for each $x \in T$, $fe = g$, $ge = g$, $he = h$; let δ be the (right) congruence on U given by the partition $\{\{f, h\}, \{g\}\}$ of U). Nor is it true of two-sided congruences, even for groups.

LEMMA 3. *Let I be any right ideal of T. Then for any $u \in U\backslash I$ such that $u\delta \subseteq U\backslash I$ we have $u\delta^* \subseteq T\backslash I$.*

Proof. We have to show, for $n = 0, 1, 2,...$, the following: If $u \in U\backslash I$, $u\delta \subseteq U\backslash I$, $t \in T$ and if $u\delta_1{}^n t$, then $t \in T\backslash I$.

For $n = 0$ the statement is trivial so let us take any integer $n \geqslant 1$ and assume the statement is true for $n - 1$. Take any $u \in U\backslash I$, $t \in T$ such that $u\delta \subseteq U\backslash I$ and $u\delta_1{}^n t$. Then for some $x \in T$ $u\delta_1 x\delta_1^{n-1}t$ whence $u\delta_1 xu^{-1}u\delta_1^{n-1}tu^{-1}u$. From Lemma 2, $xu^{-1}u \in U$ and $u\delta xu^{-1}u$ whence $xu^{-1}u \in U\backslash I$. Since $xu^{-1}u\delta_1^{n-1}tu^{-1}u$ we have from the inductive hypothesis that $tu^{-1}u \in T\backslash I$ whence $t \in T\backslash I$, since I is a right ideal of T. By induction, we have the lemma.

5. REPRESENTATIONS

Let T be an inverse semigroup. Denote by α the Preston–Vagner representation of T. Then α: $T \rightarrow \mathscr{I}_T$ and for each $t \in T$, α_t: $Tt^{-1} \rightarrow Tt$ and $x\alpha_t = xt$ for each $x \in Tt^{-1}$.

LEMMA 4. *Let γ be any right congruence on T and for each $t \in T$ define*

$$\beta_t = \{(a\gamma, b\gamma) \in (T/\gamma) \times (T/\gamma) : (x, y) \in \alpha_t \text{ for some } x \in a\gamma, y \in b\gamma\}.$$

Then $\beta_t \in \mathscr{I}_{T/\gamma}$ and the function $\beta: T \to \mathscr{I}_{T/\gamma}$ which maps each $t \in T$ to β_t is a homomorphism.

Proof. (i) Now for any pair $(a\gamma, b\gamma) \in \beta_t$ there is an element $x \in Tt^{-1}$ such that $(a\gamma, b\gamma) = (x\gamma, (xt)\gamma)$.

Take any two pairs in β_t, say $(x\gamma, (xt)\gamma)$, $(x'\gamma, (x't)\gamma)$ where $x, x' \in Tt^{-1}$. Then $x\gamma = x'\gamma$ implies $(xt)\gamma = (x't)\gamma$ (γ is a right congruence) and $(xt)\gamma = (x't)\gamma$ implies

$$x\gamma = (xtt^{-1})\gamma = (x'tt^{-1})\gamma = x'\gamma.$$

Hence $\beta_t \in \mathscr{I}_{T/\gamma}$.

(ii) Take any elements $s, t \in T$. We wish to show $\beta_s\beta_t = \beta_{st}$. Take any pair in β_{st}, say $(x\gamma, (xst)\gamma)$ where $x \in Tt^{-1}s^{-1}$. Then one easily shows that $x \in Ts^{-1}$, $xs \in Tt^{-1}$ and so $(x\gamma, (xs)\gamma) \in \beta_s$, $((xs)\gamma, (xst)\gamma) \in \beta_t$, giving $(x\gamma, (xst)\gamma) \in \beta_s\beta_t$, whence $\beta_{st} \subseteq \beta_s\beta_t$.

Conversely, take any pair in $\beta_s\beta_t$, say $(a\gamma, b\gamma)$. Then there are pairs in β_s, β_t, say $(x\gamma, (xs)\gamma)$, $(y\gamma, (yt)\gamma)$, respectively, (where $x \in Ts^{-1}$, $y \in Tt^{-1}$), such that

$$a\gamma = x\gamma, \qquad (xs)\gamma = y\gamma, \qquad (yt)\gamma = b\gamma.$$

Then

$$ys^{-1} \in (Tt^{-1})\, s^{-1} = Tt^{-1}s^{-1} = T(st)^{-1},$$
$$ys^{-1}\gamma(xs)\, s^{-1} = x,$$

and

$$(ys^{-1})(st)\ \gamma xst\gamma yt.$$

Therefore

$$(a\gamma, b\gamma) = (x\gamma, (yt)\gamma) = ((ys^{-1})\gamma, (ys^{-1}st)\gamma) \in \beta_{st}.$$

Hence $\beta_s\beta_t \subseteq \beta_{st}$, giving $\beta_s\beta_t = \beta_{st}$ as required.

Remark 2. The representation β is the one obtained if one applies the construction of [6, Theorem 1] to the representation $\rho: T \to \mathscr{T}_{T/\gamma}$, defined by, for all $x, t \in T$, $(x\gamma)\,\rho_t = (xt)\gamma$ [1, Section 11.1]. This observation is due to G. B. Preston.

Remark 3. If we put $\gamma = \mathscr{L}$ and $T = S$ then β is equivalent to the representation $\theta: S \to T_E$ of [4].

THEOREM 5. *Let U be any inverse subsemigroup of any inverse semigroup T. Then any representation of U by one-to-one partial transformations can be extended to one of T, in the following sense: for any set X and any representation $\rho: U \to \mathscr{I}_X$, there is a set Y, disjoint from X, and a representation $\sigma: T \to \mathscr{I}_{X \cup Y}$, such that for all $u \in U$, $\sigma_u \mid X = \rho_u$. It follows that $Y\sigma_u \subseteq Y$ for each $u \in U$ and so $\sigma\mid_U$ is the sum of the representations $(\sigma\mid_U) \mid X = \rho$ and $(\sigma\mid_U) \mid Y$.*

Proof. Let X be any set and let $\rho: U \to \mathscr{I}_X$ be any homomorphism. We define

$$D = \{x \in X : x \in \text{Domain } \rho_u \text{ for some } u \in U\},$$

and for each $x \in D$ we define

$$O(x) = \{y \in X : y = x\rho_u \text{ for some } u \in U\},$$

and we call $O(x)$ an *orbit* of ρ.

It follows easily from $\rho_{u^{-1}} = (\rho_u)^{-1}$ (for each $u \in U$) that if $x_1 \in O(x)$ then $O(x_1) = O(x)$, whence the orbits of ρ form a partition of D. (This is well-known [1, Theorem 7.17].) Let $\{x_i : i \in I\}$ be a transversal of the set of orbits of ρ, for some index set I.

Take a fixed element x_i from this transversal. It is clear that

$$\sigma_i = \{(a, b) \in U \times U : x_i\rho_a = x_i\rho_b\}$$

is a right congruence on U; we note that it contains the pairs $(a, b) \in U \times U$ for which $x_i\rho_a = \square = x_i\rho_b$, i.e., for which $x_i \notin \text{Domain } \rho_a$ and $x_i \notin \text{Domain } \rho_b$ (we assume $\square \notin X$). Put $\gamma_i = \sigma_i{}^*$ the right congruence on T generated by σ_i, and let $\beta^{(i)}: T \to \mathscr{I}_{T/\gamma_i}$ be the representation defined as in Lemma 4. For each $i \in I$ put $X_i = \{x_i\} \times (T/\gamma_i)$ (then the sets X_i are pairwise disjoint); define a representation $\alpha^{(i)}$ of T by one-to-one partial transformations of X_i, and equivalent to $\beta^{(i)}$, in the following obvious manner: $(x_i, t\gamma_i)\,\alpha_s^{(i)} = (x_i, (t\gamma_i)\,\beta_s^{(i)})$ for all $t, s \in T$ with $t\gamma_i \in \text{Domain } \beta_s^{(i)}$. We now define σ to be the sum of the representations $\{\alpha^{(i)} : i \in I\}$.

From the definition of σ_i and from Theorem 1 we have that

$$\begin{aligned} O(x_i) &\to U/\sigma_i, & x_i\rho_u &\mapsto u\sigma_i && (\text{where } u \in U \text{ and Domain } \rho_u \ni x_i),\\ U/\sigma_i &\to T/\gamma_i, & u\sigma_i &\mapsto u\gamma_i && (u \in U) \end{aligned}$$

and (hence)

$$O(x_i) \to X_i, \qquad x_i\rho_u \mapsto (x_i, u\gamma_i)$$

describe well-defined, one-to-one functions. We identify each element $x_i\rho_u \in O(x_i)$ with the pair $(x_i, u\gamma_i)$; further we then put

$$Y_i = X_i \backslash O(x_i) \qquad \text{and} \qquad Y = \bigcup_{i \in I} Y_i.$$

Then $X \cup Y = (\bigcup_{i\in I} X_i) \cup (X \backslash D)$ and $\sigma\colon T \to \mathscr{I}_{X\cup Y}$ is a representation. It remains to show that $\sigma_u|_X = \rho_u$ for each $u \in U$.

Take any $u \in U$. Take any element in the domain of $\sigma_u|_X$; it is first of all of the form $(x_i, a\gamma_i)$, for some $i \in I$, $a \in Tu^{-1}$; secondly, $(x_i, a\gamma_i) \in X$ if and only if $a\gamma_i = v\gamma_i$ for some $v \in U$ with $x_i \in \text{Domain}\,\rho_v$. Now $(vuu^{-1})\gamma_i = (auu^{-1})\gamma_i = a\gamma_i = v\gamma_i$ so $vuu^{-1}\sigma_i v$ (Theorem 1); i.e., $x_i\rho_{vuu^{-1}} = x_i\rho_v$. Therefore $(x_i, a\gamma_i) = (x_i, v\gamma_i) = x_i\rho_v = x_i\rho_{vuu^{-1}} = x_i\rho_{vu}\rho_{u^{-1}} \in \text{Range}\,\rho_{u^{-1}} = \text{Domain}\,\rho_u$.

Conversely, take any $x \in \text{Domain}\,\rho_u$. Then $x \in D$ so $x = x_i\rho_v$ for some $i \in I$, $v \in U$. Then $x = x\rho_u(\rho_u)^{-1} = x_i\rho_v\rho_{uu^{-1}} = x_i\rho_{vuu^{-1}} = (x_i, (vuu^{-1})\gamma_i) \in \text{Domain}(\sigma_u|_X)$ since $(vuu^{-1})\gamma_i \in \text{Domain}\,\beta_u^{(i)}$. Therefore Domain $\rho_u = \text{Domain}(\sigma_u|_X)$. Further, for the above element $x \in \text{Domain}\,\rho_u$,

$$x\sigma_u = (x_i, (vuu^{-1})\gamma_i\beta_u^{(i)}) = (x_i, (vu)\gamma_i),$$

and

$$x\rho_u = x_i\rho_v\rho_u = (x_i, (vu)\gamma_i).$$

We now have $\sigma_u|_X = \rho_u$, as required.

To prove that $Y\sigma_u \subseteq Y$ take any $y \in Y$ such that $y \in \text{Domain}\,\sigma_u$. Then $(y\sigma_u)\sigma_{u^{-1}} = y \in Y$ so $y\sigma_u \notin X$, whence $y\sigma_u \in Y$. The theorem is proved.

Remark 4. The proof of the above theorem could be shortened by appealing to the results of [1, Section 7.3], but we shall need later the details of our above proof, to obtain the strong amalgamation property. The details would not be necessary to obtain the weak amalgamation property.

COROLLARY 6. *Any representation of U by one-to-one partial transformations can be extended in the sense of the theorem to a faithful representation of T.*

Proof. Take the sum of σ and any faithful representation of T, say $\rho\colon T \to \mathscr{I}_Z$, where $(X \cup Y) \cap Z = \square$.

It is quite easy to put the theorem into the following form.

THEOREM 7. *Let U be any inverse subsemigroup of any inverse semigroup T; let X be any set and $\rho\colon U \to \mathscr{I}_X$ any representation. For any set Z such that $Z \cap X = \square$ and $|Z| \geqslant |X|\,|T|$ there is a representation $\rho'\colon T \to \mathscr{I}_{X\cup Z}$ such that $\rho_u'|_X = \rho_u$ for all $u \in U$. Consequently $Z\rho_u' \subseteq Z$ for each $u \in U$ and so $\rho'|_U$ is the sum of $(\rho'|_U)|X = \rho$ and $(\rho'|_U)|Z$.*

Proof. Let us examine the cardinality of the set Y in the proof of Theorem 5. We have $|I| \leqslant |X|$ and for each $i \in I$, $|Y_i| \leqslant |X_i| = |T/\gamma_i| \leqslant |T|$. Since $Y = \bigcup_{i\in I} Y_i$ we have $|Y| \leqslant |X|\,|T| \leqslant |Z|$. Any one-to-one function from $X \cup Y$ into $X \cup Z$ which maps the elements of X identically, can be used to obtain a representation ρ': $T \to \mathscr{I}_{X \cup Z}$ equivalent to σ and with the required properties.

6. Strong Amalgamation

THEOREM 8. *Inverse semigroups have the strong amalgamation property.*

Proof. Let S and T be any inverse semigroups with a common inverse subsemigroup U. We assume without loss of generality that $S \cap T = U$.

Let A be any infinite set such that $|A| \geqslant |S|$, $|A| \geqslant |T|$. We shall show that there are faithful representations P: $S \to \mathscr{I}_A$, Σ: $T \to \mathscr{I}_A$ such that $P|_U = \Sigma|_U$ and $(SP) \cap (T\Sigma) = UP(= U\Sigma)$.

Form an enumerated partition $\{A_i: i = 1, 2, 3, ...\}$ of A such that $|A_i| = |A|$ for each i. Further partition each A_i into two subsets B_i, C_i, such that $|B_i| = |C_i| = |A_i|$. Then

$$A = B_1 \cup C_1 \cup B_2 \cup C_2 \cup \cdots.$$

Since we will be using Theorem 7 many times, we note here that for any subset A' of A, and for $i = 1, 2, 3, ...,$

$$|A'|\,|S| \leqslant |A| = |B_i| = |C_i|$$

and

$$|A'|\,|T| \leqslant |A| = |B_i| = |C_i|.$$

Since $|S| \leqslant |B_1|$ and $|T\backslash U| \leqslant |C_1|$ we can (by identifying elements) assume that $S \subseteq B_1$, $T\backslash U \subseteq C_1$, and $|C_1\backslash T| = |C_1|$.

For convenience we shall now denote the symmetric inverse semigroup on a set X by $\mathscr{I}(X)$, instead of $\mathscr{I}_X$.

Let $\rho^{(1)}$: $S \to \mathscr{I}(S) \leqslant \mathscr{I}(B_1)$ and σ': $T \to \mathscr{I}(T) \leqslant \mathscr{I}(U \cup C_1)$ be the Preston–Vagner representations of S and T, respectively. Then $\rho^{(1)}|_U$ is the sum of $(\rho^{(1)}|_U) \mid U$ and $(\rho^{(1)}|_U) \mid (S\backslash U)$ (easily shown); also $(\sigma'|_U) \mid U = (\rho^{(1)} \mid {}_U) \mid U$. Now put $X = S\backslash U$, $\rho = (\rho^{(1)}|_U) \mid (S\backslash U)$ and construct the representation σ: $T \to \mathscr{I}(X \cup Y)$ as in the proof of Theorem 5, with the further condition that the transversal element of an orbit of ρ is chosen to be an idempotent if the orbit contains an idempotent; since $|Y| \leqslant |X|\,|T| \leqslant |A| = |C_1\backslash T|$ we can assume without loss of generality that $Y \subseteq C_1\backslash T$. Let $\sigma^{(1)}$: $T \to \mathscr{I}(B_1 \cup C_1)$ be the sum of σ and σ'. Then $\rho^{(1)}|_U = (\sigma^{(1)}|_U) \mid B_1$.

Now the representation $(\sigma^{(1)} \mid {}_U) \mid C_1$: $U \to \mathscr{I}(C_1)$ can be extended (by Theorem 7) to a representation $\rho^{(2)}$: $S \to \mathscr{I}(C_1 \cup B_2)$. Then $(\rho^{(2)}|_U) \mid B_2$: $U \to \mathscr{I}(B_2)$ is extended to a representation $\sigma^{(2)}$: $T \to \mathscr{I}(B_2 \cup C_2)$ and then $(\sigma^{(2)}|_U) \mid C_2$: $U \to \mathscr{I}(C_2)$ is extended to a representation

$\rho^{(3)}: S \to \mathscr{I}(C_2 \cup B_3)$. This process is continued so that we obtain, for $i = 1, 2, 3, \ldots$, representations

$$\rho^{(i)}: S \to \mathscr{I}(C_{i-1} \cup B_i) \qquad \text{(where we put } C_0 = \Box\text{)},$$
$$\sigma^{(i)}: T \to \mathscr{I}(B_i \cup C_i)$$

such that

$$(\rho^{(i)}|_U) \mid B_i = (\sigma^{(i)}|_U) \mid B_i,$$
$$(\rho^{(i+1)}|_U) \mid C_i = (\sigma^{(i)}|_U) \mid C_i.$$

We now let P be the sum of $\{\rho^{(i)}: i = 1, 2, \ldots\}$ and let Σ be the sum of $\{\sigma^{(i)}: i = 1, 2, \ldots\}$. Since $\rho^{(1)}$ and $\sigma^{(1)}$ are faithful, P and Σ are faithful. Further, it is easily seen that $P|_U = \Sigma|_U$.

Put $(SP) \cap (T\Sigma) = V$, an inverse subsemigroup of $\mathscr{I}(A)$. Then $UP = U\Sigma \subseteq V$. Since P and Σ are monomorphisms we have that $(P^{-1}|_V): V \to VP^{-1}$ and $(\Sigma^{-1}|_V): V \to V\Sigma^{-1}$ are isomorphisms and so $(P|_{(VP^{-1})}) \circ (\Sigma^{-1}|_V) = \phi$ say, is an isomorphism between VP^{-1} and $V\Sigma^{-1}$. For each $a \in VP^{-1}$ we write $a\phi = a'$; then for each $u \in U \subseteq VP^{-1}$, $u = u'$. We have to show that $V \subseteq UP$, i.e., that $VP^{-1} \subseteq U$, so let us assume for the moment that there is an element $a \in (VP^{-1})\backslash U$. One case is dealt with easily.

Case I, $aa^{-1} \in U$. Then $a'(a')^{-1} = aa^{-1}$ and $aa^{-1}P_a = aa^{-1}\rho_a^{(1)} = a$, $a'(a')^{-1}\Sigma_{a'} = a'(a')^{-1}\sigma_a^{(1)} = a'$; but $a \neq a'$ so $P_a \neq \Sigma_{a'}$, a contradiction to the fact that $a \in VP^{-1}$.

Case II, $aa^{-1} \notin U$. Put $e = aa^{-1}$. Then $e' = a'(a')^{-1}$. We show that $P_e \neq \Sigma_{e'}$ (from which it follows that $P_a \neq \Sigma_{a'}$) by showing that $e \notin \text{Domain } \Sigma_{e'}$ (i.e., that $e \notin \text{Domain } \sigma_{e'}$); note that $e \in \text{Domain } \rho_e^{(1)} \subseteq \text{Domain } P_e$.

Since each orbit of $\rho = (\rho^{(1)}|_U) \mid (S\backslash U)$ is contained in an $\mathscr{R}$-class of S an orbit contains at most one idempotent. Hence either $e \in X\backslash D$, whence $e \notin \text{Domain } \sigma_{e'}$, or $e = x_i$, for some $i \in I$. Assume the latter case and write $\delta_i = \delta$, $\gamma_i = \gamma$. Then

$$\delta = \{(a, b) \in U \times U: e\rho_a^{(1)} = e\rho_b^{(1)}\}$$

and γ is the right congruence on T generated by δ.

Suppose, for the moment, that $e \in \text{Domain } \sigma_{e'}$; then $e \in \text{Domain } \alpha_{e'}^{(i)}$. Hence for some $u \in U$, $t \in Te'$,

$$e = (e, u\gamma) = (e, t\gamma).$$

Thus $e\rho_u^{(1)} = e$, $u\gamma = t\gamma$ and $t \in Te'$. We show next that $t = e'$.

As above let $\sigma': T \to \mathscr{I}(T)$ be the Preston–Vagner representation of T. Put

$$\xi = \{(a, b) \in T \times T: e'\sigma_a' = e'\sigma_b'\},$$

a right congruence on T. Because of the isomorphism $\phi: VP^{-1} \to V\Sigma^{-1}$ we can show that $\xi \cap (U \times U) = \delta$ (whence $\gamma \subseteq \xi$) as follows. Take any $a, b \in U$. Then

$$\begin{aligned}
&(a, b) \in \delta \\
&\quad \Leftrightarrow e\rho_a^{(1)} = e\rho_b^{(1)} \\
&\quad \Leftrightarrow (e \in Saa^{-1} \cap Sbb^{-1} \text{ and } ea = eb) \text{ or } (e \notin Saa^{-1} \cup Sbb^{-1}) \\
&\quad \Leftrightarrow (e \leqslant aa^{-1}, e \leqslant bb^{-1} \text{ and } ea = eb) \text{ or } (e \nleqslant aa^{-1} \text{ and } e \nleqslant bb^{-1}) \\
&\quad \Leftrightarrow (e' \leqslant aa^{-1}, e' \leqslant bb^{-1} \text{ and } e'a = e'b) \text{ or } (e' \nleqslant aa^{-1} \text{ and } e' \nleqslant bb^{-1}) \\
&\quad \Leftrightarrow e'\sigma_a' = e'\sigma_b' \\
&\quad \Leftrightarrow (a, b) \in \xi.
\end{aligned}$$

From $\gamma \subseteq \xi$ and $u\gamma = t\gamma$ we have $(u, t) \in \xi$, i.e., $e'\sigma_u' = e'\sigma_t'$. Since $e\rho_u^{(1)} = e$ we have $e \leqslant uu^{-1}$ and $eu = e$ whence $e' \leqslant (uu^{-1})' = uu^{-1}$ and $e'u = e'$, i.e., $e'\sigma_u' = e'$. Hence $e' = e'\sigma_t' = e't$.

We now have $te' = t$ and $e't = e'$; consequently $e'\mathscr{L}t$ (in T). Also

$$t^2 = (te')t = t(e't) = te' = t,$$

whence $t = e'$.

Put $I = \bigcup \{R \in T/\mathscr{R}: R \ngtr R_{e'}\}$, a right ideal of T. Then $e' \in I$ and $u\gamma e'$. Hence by Lemma 3 we will have a contradiction when we show $u\delta \subseteq U\backslash I$.

Take any element $v \in U$. We have $e \in \text{Domain } \rho_v^{(1)} = Svv^{-1}$ if and only if $e < vv^{-1}$ ($e \neq vv^{-1} \in U$), which holds if and only if $e' < (vv^{-1})' = vv^{-1}$ and this latter statement is equivalent to $R_{e'} < R_v$ (in T), i.e., to $v \in T\backslash I$. Hence $e\rho_v^{(1)} \neq \square$ if and only if $v \in U\backslash I$ and so $\delta \subseteq ((U\backslash I) \times (U\backslash I)) \cup (I \times I)$. In particular $u \in U\backslash I$ and $u\delta \subseteq U\backslash I$, as required.

Remark 5. As corollaries to Theorem 8 we have that the class of groups and the class of semilattices have the strong amalgamation property. It is rather surprising then that the class of semilattices of groups does not have even the weak amalgamation property (e.g., put $S = G$, a simple group containing a subgroup U with a proper normal subgroup N, and put $T = U \cup (U/N)$ with $u_1(Nu_2) = Nu_1u_2$ and $(Nu_2)\, u_1 = Nu_2u_1$ for any $u_1 \in U$, $Nu_2 \in U/N$).

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